San Diego Math Circle
Euler Class
02/07/2009

Classical Geometry Constructions

In classical geometry (that is, with a compass and straight edge), there are four axioms that describe the basic building blocks for all possible constructions. These are:

$C1$. Given a point, a line can be drawn through the point.

$C2$. Given two points, a line can be drawn through both points.

$C3$. Given a point, a circle can be drawn centered at that point.

$C4$. Given a point p and a segment of length r, a circle can be drawn with radius r centered at p.

$C5$. Given two points, a circle can be drawn using one point as the center and going through the other point.

From these axioms, there are a few more complicated procedures that are useful to know:

- Given a line segment, its perpendicular bisector can be drawn.
- Given an angle, its angle bisector can be drawn.

Combining the axioms with geometric knowledge, many constructions are possible. However, two constructions are impossible under the constraints of using a straight edge and compass:

- Given an angle, construct its angle trisector.
- Given a line segment s that is the edge of a cube, construct a line segment for a cube with twice the volume. (that is, find $s\sqrt[3]{2}$)

Origami

Origami construction is similar to classical geometry construction, but with a different set of constraints. Here, we assume that a piece of paper is infinitely large and flat, and that we can fold with unlimited precision. In origami construction, the axioms are:

$O1$. Given two points, a line can be folded through both points.

$O2$. Given two points, p_1 and p_2, it is possible to fold p_1 onto p_2.

$O3$. Given two lines, l_1 and l_2, it is possible to fold l_1 onto l_2.

$O4$. Given a point, p, and a line, l, it is possible to fold a line perpendicular to l passing through p.

$O5$. Given two points, p_1 and p_2, and a line, l, it is possible to fold p_1 onto l such that the creases passes through p_2.

$O6$. Given two points, p_1 and p_2, and two lines, l_1 and l_2, it is possible to fold p_1 onto l_1 and p_2 onto l_2.

$O7$. Given a point, p, and two lines l_1 and l_2, it is possible to fold a line perpendicular to l_2 that places p onto l_1.

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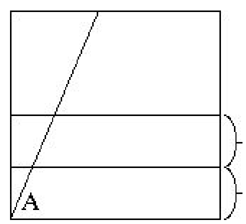
Exercises

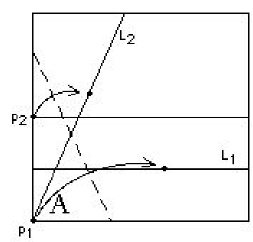
E1. First, fold your paper in half and draw a point, p_1 about an inch away from the bottom edge. Let the bottom edge of your paper be l_1. Now draw a point, p_2 on the left or right edge of your paper. Fold axiom $O5$. Repeat for many different p_2. What is the resulting shape formed by the creases? Why is this the case?

Trisecting an angle

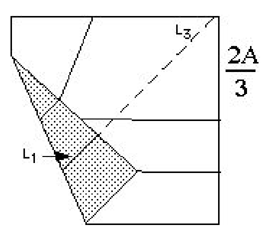
1. First, let the angle to be trisected be acute, and orient it in the lower left corner. Make two parallel, equidistant horizontal creases at the bottom.



2. Using axiom $O6$, fold p_1 onto l_1 and p_2 onto l_2.



3. Extend the former crease at l_1 all the way through to from l_3. Unfold and extend l_3 to the corner: it should intersect and form an angle that is $2/3$ the original angle.



Exercises

E2. Show the angle was trisected.

E3. Extend the method to trisecting obtuse angles.

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Other Problems

The **Fold and Cut** Problem: Given a piece of paper, fold it flat however you wish, and make one complete straight cut. What shapes are possible (given an infinitely foldable piece of paper)?

References

Note: you should be able to click on the URL's and be taken directly to the website.

- Thomas Hull has a page on origami and geometric construction at `http://kahuna.merrimack.edu/~thull/omfiles/geoconst.html` Some of the figures in this document and solutions to the exercises can be found there.
- Erik Demaine has a page on various folding topics, including the **Fold and Cut** problem at `http://erikdemaine.org/folding/`. Here is a direct link to the **Fold and Cut** examples: `http://erikdemaine.org/foldcut/examples/`. There are also some fun folding puzzles here: `http://erikdemaine.org/puzzles/` for you to print-out and enjoy.